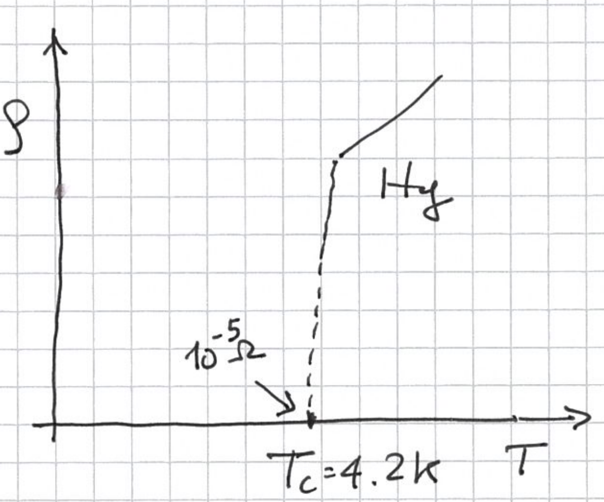
1. FENOMENOLOGIA DELLA SUPERCONDUTTIVITÀ

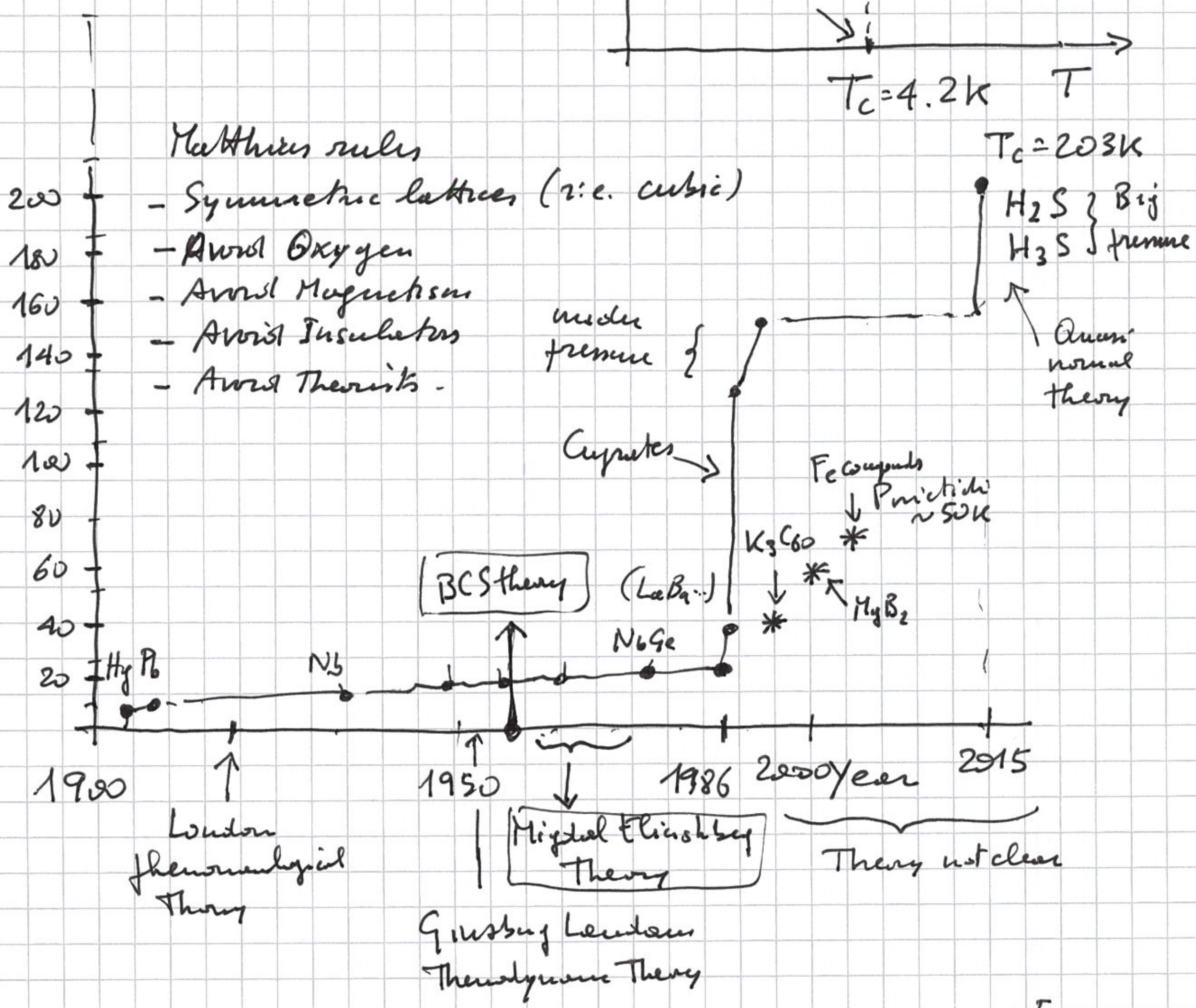
04/10/2015 (2)

Phenomenology of Superconductors

K. Onnes 1911
(1908 Liquid He) $T_c = 1.5K$

- $SC \neq$ perfect conductor
- Macroscopic quantum state
- Difficult to understand



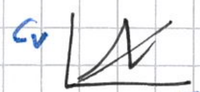


- Zero resistance (electrical) - Persistent current $> 10^5$ years
- Absence of thermoelectric effects (Seebeck, Peltier, Thomson Heat)
- Ideal diamagnetism ($\chi_m = -1$). Weak magnetic fields are completely screened away (~~expelled~~) from a bulk SC
- Meissner effect. Cooling below T_c in a magnetic field H the M. field is completely expelled from the bulk SC.

- Flux quantization - Magnetic flux through a SC ring is quantized and constant in time - Predicted by London 1950 and verified in 1961.

- Isotope effect

  $T_c \simeq M_{ion}^{-1/2}$ evidence of el-phonon interaction.

- Specific Heat 

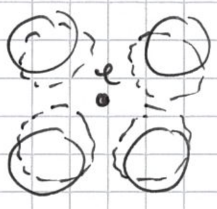
- Magnetic field destroys SC

Real phase transition

B Theoretical considerations:

BCS: quantum mechanical treatment of SC was done in 1957, 46 years after the discovery of SC. But even BCS theory did not clarify the nature of the interaction el-ph because it does not include retardation etc

In general electrons repel due to Q and the attraction due to lattice polarization is very small.



Also the appearance of BCS theory did not lead to the finding of better materials.

Phenomenological Theories

Assumption of a coherent SC state without giving the origin

London, Pippard, Landau, Ginzburg

could lead to important progress and explained and predicted a number of properties.

18
~~20~~/10/2015

4

THE PUZZLE OF SUPERCONDUCTIVITY

- Problem appear relatively simple to describe
  General property of many metals (not all)
- Coulomb interaction $\simeq 1\,eV/atom$
  SC energetic $T_C \to 10^{-8}\,eV/atom$

$$\frac{e^2}{r=1\text{Å}} = 15\,eV \qquad \varepsilon(q,\omega) \qquad 1\,eV \leftrightarrow 11.000\,K$$

$T_c \sim 1-20°K$

J. Bardeen's view:

Quantum theory

→ **Normal Metals**: Sommerfeld-Bloch, Fermi Liquid, Landau et al

→ electron-phonon interaction; Isotope effect $T_c \sim M^{-1/2}$

→ Coulomb Interaction, Heisenberg

electron-phonon interaction → **ENERGY GAP MODEL**

ENERGY GAP MODEL → Pippard Theory ← London Theory ← E.M. properties: $E=0$; $B=0$, Meissner effect

ENERGY GAP MODEL → Two fluid model, Gorter-Casimir, Landau Ginzburg ← Thermal properties: Specific Heat, 2nd order transition, Thermal Conductivity, Ultrasonic Attenuation

Failed theories of SC:

Thompson; Einstein, Bohr, Brillouin, Kronig, F. Bloch, Landau, Heisenberg, M. Born, Feynman

1911 - 1957 → 46 years

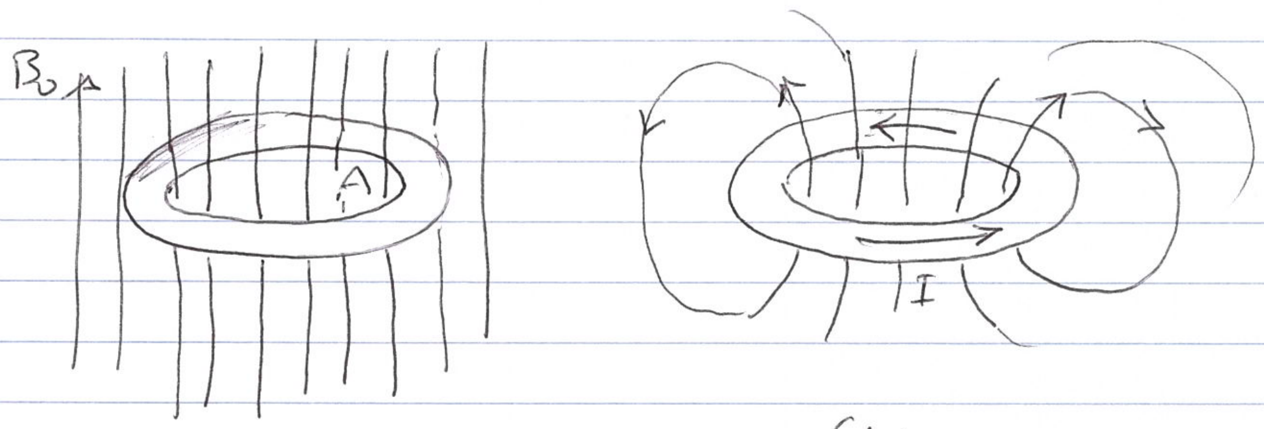
- Development of BCS theory is among the most outstanding intellectual achievements in theoretical physics
- Theories emerge in a very different way from what is presented in classrooms or textbooks.

Key points of the puzzle:

- Real phase transition - new equilibrium phase - ordered
- Zero resistance but not perfect metal
  scattering must go to zero abruptly (London)
- Meissner effect - expulsion of B
  related questions gauge invariance / Wavefunction rigidity
- Gap in the one electron excitations
  J. Bardeen - distortion of lattice → gap - (one electron no M.)
- Thermal properties. Sp. Heat (real phase transition)
  Ultrasonic attenuation, Spin resonance etc.
- Phenomenological approaches, London, Pippard, Landau Ginzburg.
- Isotope effect $T_c \sim M^{-1/2}$. A basic hint
- BUT perturbative theories would not lead to Meissner
- Cooper pair - breakthrough - ∃ of a bound state for any weak interaction due to Fermi surface.
- Development of BCS theory
- Extensions BCS - BEC & Migdal-Eliashberg G. Function
- New developments in materials.

Persistent currents lead to a constant magnetic flux

Ideal conductor.



(a) (b)

1. $T > T_c$ apply a magnetic field
2. Cool to the ideal conductor state $T < T_c$ keeping the magnetic field applied
3. Change the magnetic field (i.e. $B = 0$)
   Induced currents compensate this change and the magnetic flux remains constant (Lenz's law)

Lenz's law: a change in the magnetic field induces a current that compensates for the flux change due to the applied field

Faraday: induced emf in the ring $\quad -\frac{d\phi}{dt} = -A\frac{d(B - B_0)}{dt}$

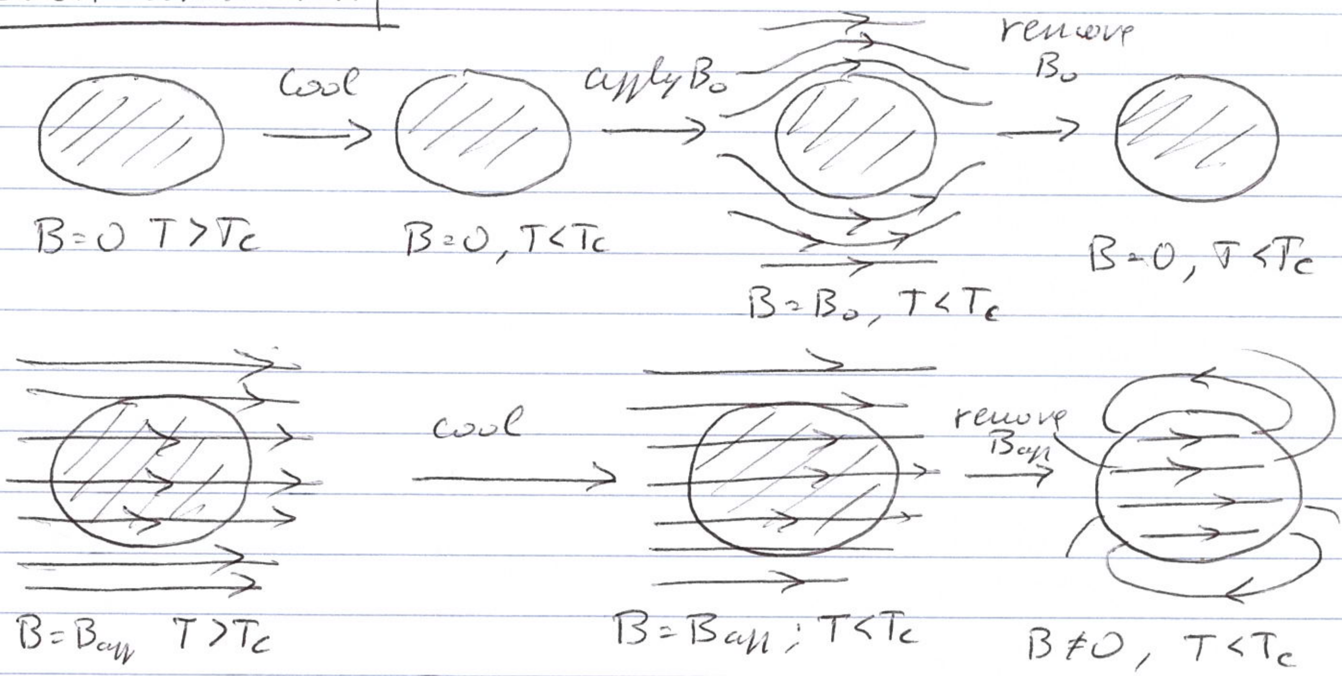
$\rightarrow$ Induced current $\quad \underset{\uparrow \text{ self inductance}}{L}\frac{dI}{dt} = -A\frac{dB}{dt}$ (no ohmic term IR)

Integrating $\rightarrow$ $\quad LI + BA = \text{const}$

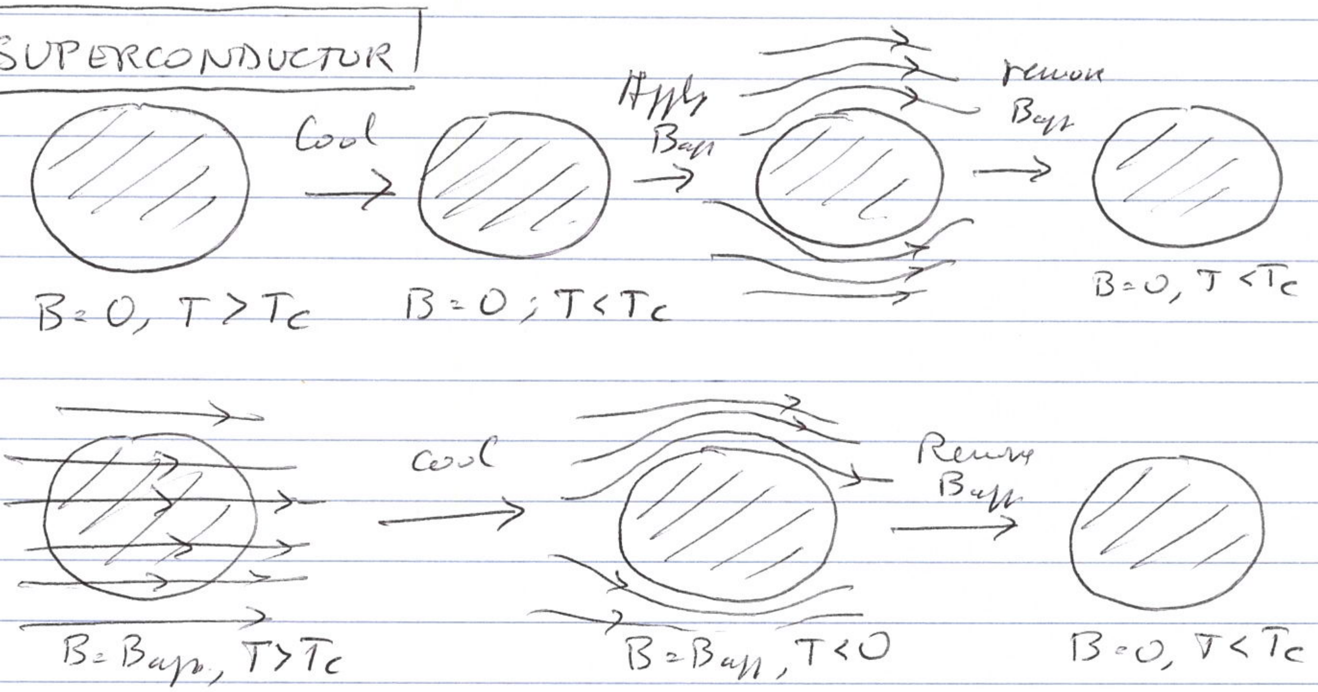
$\phi = LI$ (Induttanza elettrica)

Meissner Effect

IDEAL CONDUCTOR



SUPERCONDUCTOR



Two fluid model

Standard electrons (resistive)

$$\vec{J}_n = -n_n e \langle v_n \rangle = \frac{n_n e^2 \tau}{m} \vec{E} \quad \text{(Drude law)}$$

Free electrons

$$m \frac{dV_s}{dt} = -e\vec{E} \quad ; \quad J_s = -n_s e V_s$$

$$\frac{\partial J_s}{\partial t} = n_s e \frac{\partial V_s}{\partial t} = \frac{n_e e^2}{m} E$$

For a constant current flowing $\frac{\partial J_s}{\partial t} = 0 \rightarrow E = 0$

$\hookrightarrow J_n = 0$

→ All of the steady current is carried by the free electrons

Magnetic field in a perfect conductor

An ideal conductor does not ~~expell~~ expel the magnetic field but keeps it constant.

From Maxwell's eqs.:

Faraday's law $\quad \nabla \times E = -\frac{\partial B}{\partial t}$

using the relation between J and E in a p.c.

$$\nabla \times \frac{\partial J_{pc}}{\partial t} = -\frac{n_{pc} e^2}{m} \frac{\partial B}{\partial t}$$

Ampere - Maxwell $\quad \nabla \times H = J_f + \frac{\partial D}{\partial t}$

$\mu \simeq 1 \rightarrow H \approx \frac{B}{\mu_0}$ and also $\frac{\partial D}{\partial t} \simeq 0$

$\rightarrow$ Ampere's law $\quad \nabla \times B = \mu_0 J_{pc} \quad (J_f \simeq J_{pc})$

and eliminating J_{pc} from previous equation

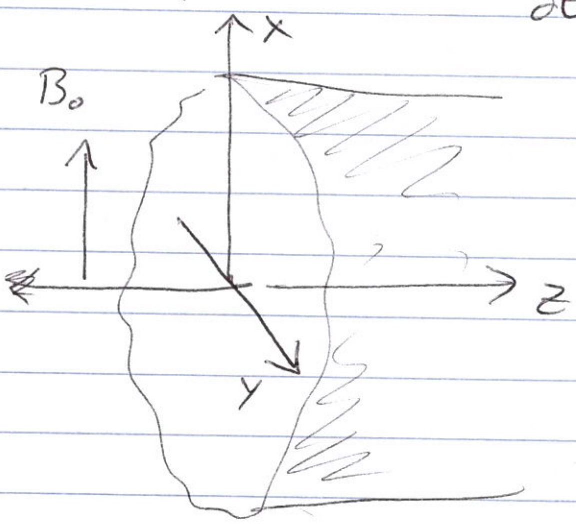
$$\nabla \times \left(\nabla \frac{\partial B}{\partial t}\right) = -\frac{\mu_0 n_{pc} e^2}{m} \frac{\partial B}{\partial t}$$

$$\nabla \times (\nabla B) = \text{grad}(\text{div}\, B) - \nabla^2 B$$

but $\text{div}\, B = 0$ (no magnetic charges)

$$\boxed{\nabla^2 \left(\frac{\partial B}{\partial t}\right) = \frac{\mu_0 n_{cp} e^2}{m} \frac{\partial B}{\partial t}}$$

This equation defines $\frac{\partial B}{\partial t}$ in an ideal conductor



Consider a uniform field outside the conductor

$$B_0 = B_0 e_x$$

$$\frac{\partial^2}{\partial z^2} \left(\frac{\partial B_x(z,t)}{\partial t}\right) = \frac{1}{\lambda_{pc}^2} \frac{\partial B_x(z,t)}{\partial t}$$

$$\frac{\mu_0 n_{pc} e^2}{m} = \frac{1}{\lambda_{pc}^2}$$

General solution

$$\frac{\partial B_{x}(z,t)}{\partial t} = a(t)\, e^{-z/\lambda_{pc}} + b(t)\, e^{+z/\lambda_{pc}}$$

$b = 0$

at $z = 0 \rightarrow a = \frac{\partial B_0}{\partial t}$

$$\boxed{\frac{\partial B_{x}(z,t)}{\partial t} = \frac{\partial B_0(t)}{\partial t}\, e^{-z/\lambda_{pc}}}$$

So any change in the magnetic field is attenuated exponentially. And λ_{pc} is the characteristic length of this attenuation.

<u>NB</u>: This does not mean that the magnetic field must be <u>expelled</u>

Flux expulsion implies $B = 0$

Here we have instead constant $B \rightarrow \frac{\partial B}{\partial t} = 0$

How to modify the description of a perfect conductor to get the properties of a SC which imply $B = 0$?

The London equations

After the discovery of the Meissner effect F & H London proposed a phenomenological relation the describes the expulsion of the magnetic field and the penetration depth at the surface.
We have seen that for a perfect conductor we have

$$\nabla \times \left(\frac{\partial J_{pc}}{\partial t} \right) = - \frac{n_{pc} e^2}{m} \frac{\partial B}{\partial t}$$

which leads to $\frac{\partial B}{\partial t} = 0$, the magnetic field is frozen. In a SC instead we have $B = 0$, more restrictive.

So they propose to substitute the eq. for PC with the more restrictive one for the SC:

$$\boxed{\nabla \times J_s = - \frac{n_e e^2}{m} B}$$

which is associated to the other equation

$$\boxed{\frac{\partial J_s}{\partial t} = \frac{n_e e^2}{m} E}$$

London equations

They are not an explaination of SC but, together with Maxwell's equations they show that this restriction reproduces the properties of SC, in particular Meissner.
(Analogous ~~the~~ to Ohm's law for electrical resistivity)

Then procede in analogy with the study of the perfect conductor.

Consider Ampere's law

$$\nabla \times B = \mu_0 J_s$$

to substitute J_s in previous equation

$$\nabla \times (\nabla \times B) = -\frac{\mu_0 n_s e^2}{m} B = -\frac{1}{\lambda^2} B$$

with $\lambda = \left(\frac{m}{\mu_0 n e^2}\right)^{1/2}$

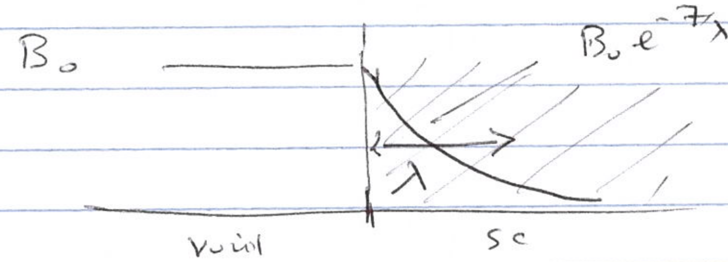
but $\nabla \times (\nabla B) = \text{grad}(\text{div } B) - \nabla^2 B = -\nabla^2 B$
since $\text{div } B = 0$

$$\nabla^2 B = \frac{1}{\lambda^2} B$$

Analogous to that of PC but $\frac{\partial B}{\partial t}$ is now substituted with B.

The only solution for a spatially uniform B is $B = 0$
And for the surface

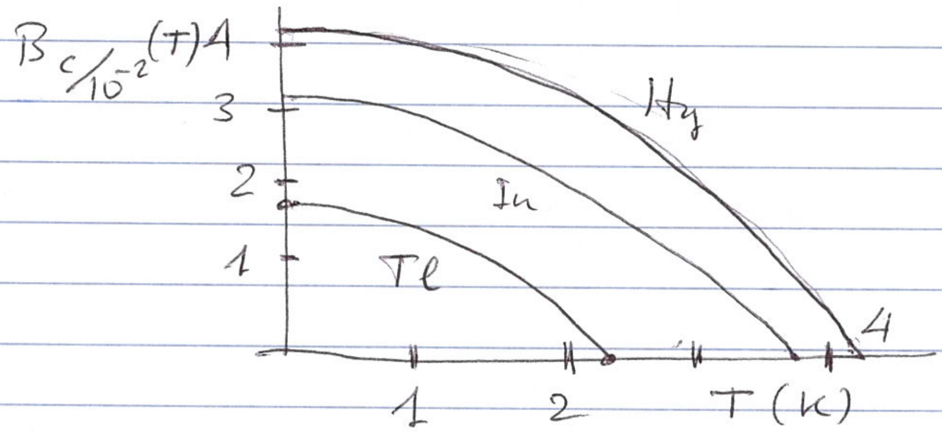
$$B_x(z) = B_0 e^{-z/\lambda}$$



Critical Magnetic Field

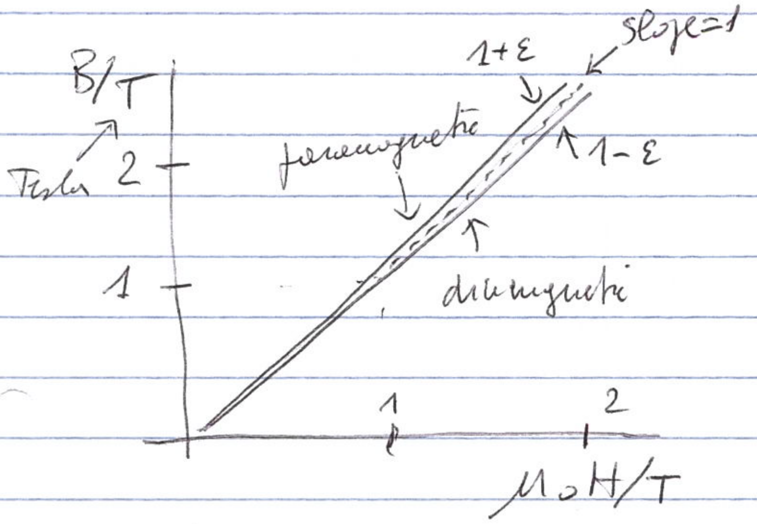
If a large field (B) is applied there is a transition from SC to normal metal.
Simplest situation:
In general it depends on the shape of the sample and the direction of the applied field.
For a cylinder with applied field // to the cylinder we observe a reversible phase transition at a critical magnetic field $B_c(T)$.

$$B_c(T) = B_0(T)\left[1 - (T/T_c)^2\right]$$



	T_c/K	$B_0(0)$/mT
aluminium	1.2	10
cadmium	0.52	2.8
indium	3.4	28
mercury	4.2	41
Zinc	0.85	5.4

Compare magnetic behavior of SC with normal effect of the magnetic field in various materials

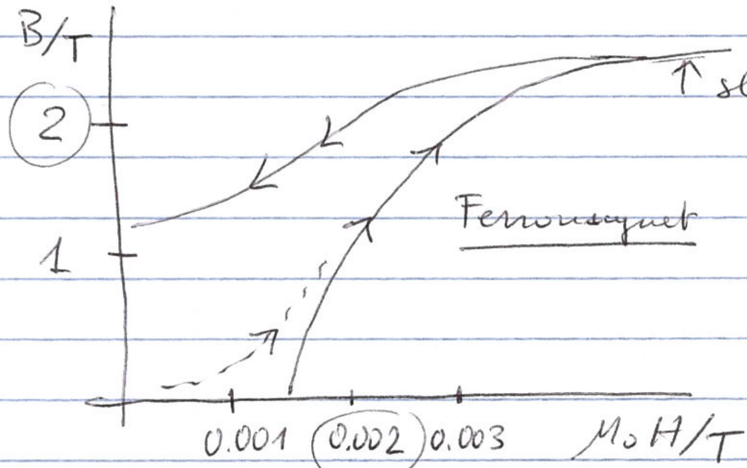


$\mu_0 H$ applied field (in Tesla)

B internal field (in Tesla)

$$B = \mu \mu_0 H$$

diamagnetic $\mu = 0.9998$
ferromagnetic $\mu = 1 + \varepsilon$

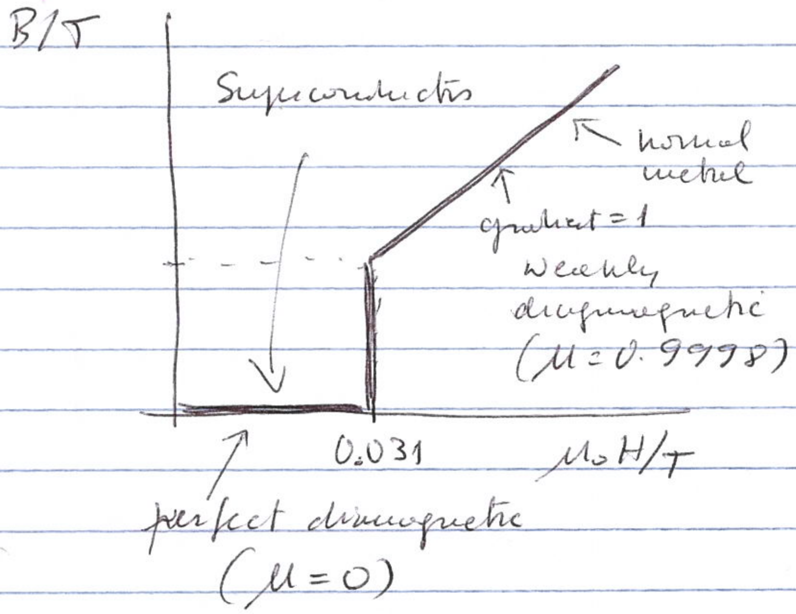


Ferromagnet

small applied field leads to large internal field → correlation

Ising like transition

Hysteresis: first order transition



Critical (II^{nd} order transition)

No hysteresis

Critical current

Current density in a normal wire is uniform over its cross section
(The value of B increases linearly with the distance from the center).
But in a SC the value of $B = 0$ in the bulk.
The field just outside the wire is $\mu_0 I/(2\pi a)$ (a = radius)
If the current is increased this magnetic field may become larger than B_c.

Critical current: $$I_c = \frac{2\pi a B_c}{\mu_0}$$

Note $I_c \propto a$

Therefore also I_c will depend on T

In usual applications $T \lesssim \frac{1}{2} T_c$ and $I_c \gtrsim 0.75\, I_c^{MAX}$

NB: In a SC the current actually flows in a thin layer at the surface.
This layer is not infinitesimal but given by the penetration depth (as we have seen).
However I_c' at the surface is $\gg I_c/\pi a^2$ because it flows only in a thin layer.

Penetration depth

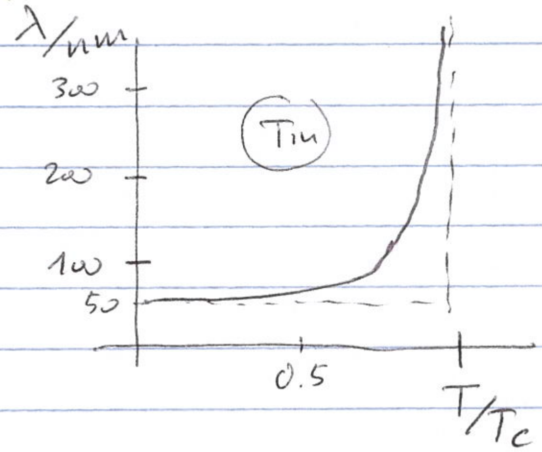
The penetration depth (of the magnetic field) depends on the density of SC electrons n_s
If all el. are SC

$$\lambda = \left(\frac{m}{\mu_0 n_s e^2}\right)^{1/2} = 1.7 \times 10^{-8}\,m \simeq 20\,nm$$

The small value of λ implies that it is difficult to measure it. → large surface/volume ratio.
Thin films, thin wires, colloidal particles etc.

$$n_s(T) \rightarrow \lambda(T)$$

For $T \ll T_c$ $\quad n_s \simeq n_s^{total}$
Near the critical temperature $\lambda \propto n_s^{-1/2}$ according to the London model



$$\lambda(T) = \frac{\lambda(0)}{\sqrt{1-(T/T_c)^4}}$$

The screening current

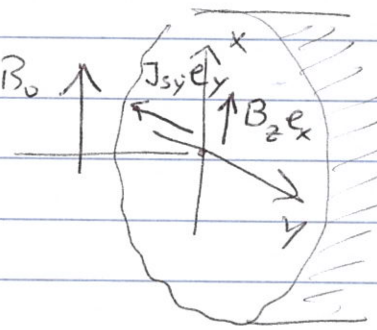
We have derived the London eqs. by eliminating J_s in the two equations.
We can instead eliminate B to obtain

$$\nabla^2 J_s = \frac{1}{\lambda^2} J_s$$

So for a semiinfinite surface we have

$$J_s(z) = J_0 e^{-z/\lambda}$$

Using Ampere's law $\nabla \times B = \mu_0 J_s$ $\quad (\bar{B} = B_x(z)\, \varepsilon_x)$



$$\nabla \times B = \frac{\partial B_x(z)}{\partial z} e_y$$

Then $J_s = \frac{1}{\mu_0} \nabla \times B =$

$$= \frac{1}{\mu_0} \frac{\partial B_x(z)}{\partial z} e_y = J_{sy}(z)\, e_y$$

where $J_{sy}(z) = \frac{1}{\mu_0} \frac{\partial B_x(z)}{\partial z}$

But we know that $B_x(z) = B_0 e^{-z/\lambda}$ so we get

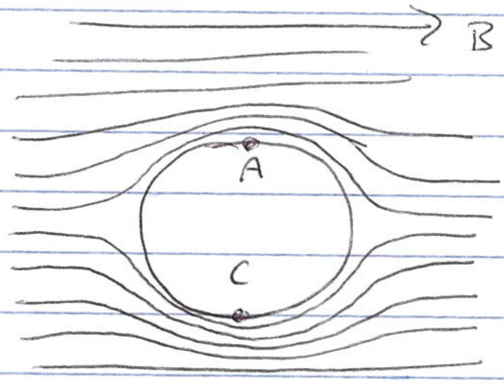
$$J_{sy}(z) = -\frac{B_0}{\mu_0 \lambda} e^{-z/\lambda}$$

Therefore the current that screens the interior of a SC from an applied field flows in a thin surface layer (λ) and the current flows // to the surface and $\perp$ to B.
London local model is good at $\xi \ll \lambda$ where ξ is the coherence length – distance over which the value of n_s changes. For $T \lesssim T_c$ London is better.

Two types of SC : Type I and Type II

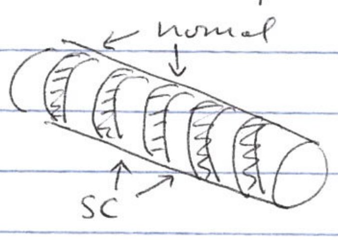
All pure elemental SC are Type I except Nb, Vanadium, Technetium.
SC alloys and in general SC with high T_c are of Type II → practical applications.

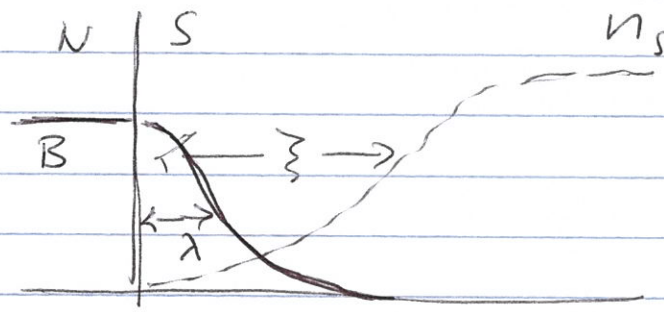
The properties described until now refer mostly to Type I



Field at A and C points is twice B.
Therefore for this shape $B_0 = \frac{1}{2} B_c$ but only at A and B

If this would destroy completely the SC state then the material would be in the normal state in the interior with a low field $B_c/2$ which is not possible
<u>This situation leads to heterogeneity</u> ; <u>Intermediate State</u>
In the range $B_c/2 < B_0 < B_c$ the material splits into slices of normal and SC material.

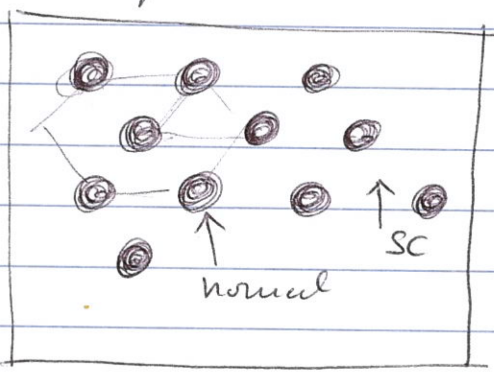


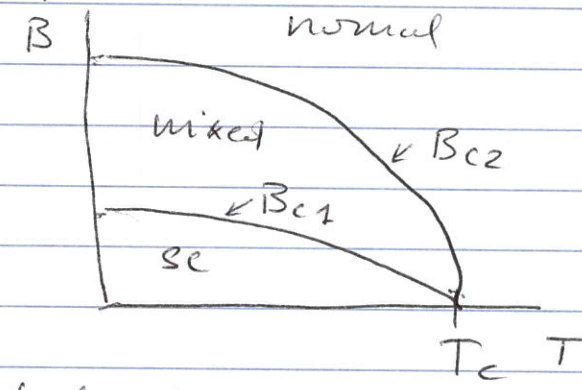


Type II SC

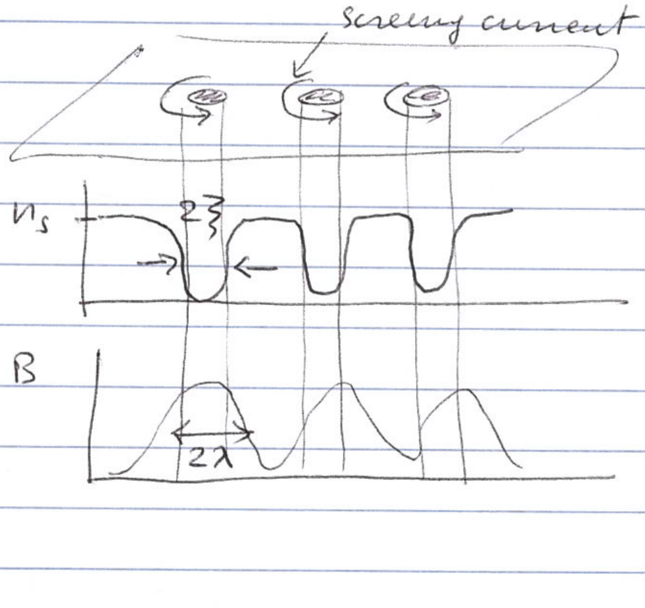
1957 Abrikosov predicted the ∃ of a different type of SC

Experiment: small magnetic particles on the surface of a SC.





Below B_{c1} the magnetic field is excluded from the bulk.
Above B_{c1} thin cylindrical regions of normal state appear - normal core - triangular lattice.
Increasing B → more closely packed.
B_{c2}: upper critical field above which all is normal
B_{c1} and B_{c2} are T dependent



$$\begin{cases} \text{Type I} & \xi > \lambda \\ \text{Type II} & \xi < \lambda \end{cases}$$

Type I: $\begin{cases} \xi \simeq 1\,\mu m \\ \lambda \simeq 50\,nm \end{cases}$

Type II Alloy Nb_3Sn $\begin{cases} \xi \simeq 3.5\,nm \\ \lambda = 80\,nm \end{cases}$

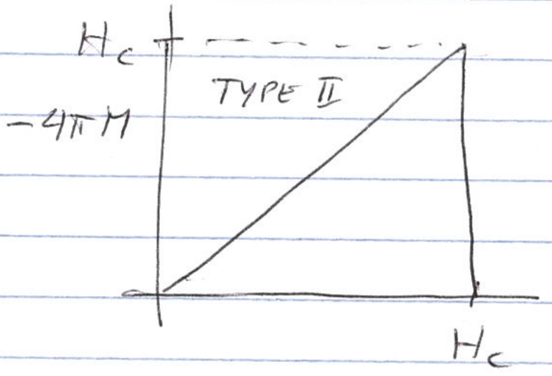
When $\xi > \lambda$ the surface energy is >0, while it is <0 when $\xi < \lambda$.
If $E_S > 0$ the system minimizes the boundary. The opposite case favours the formation of multi boundaries $\rightarrow$ symmetry breaking.

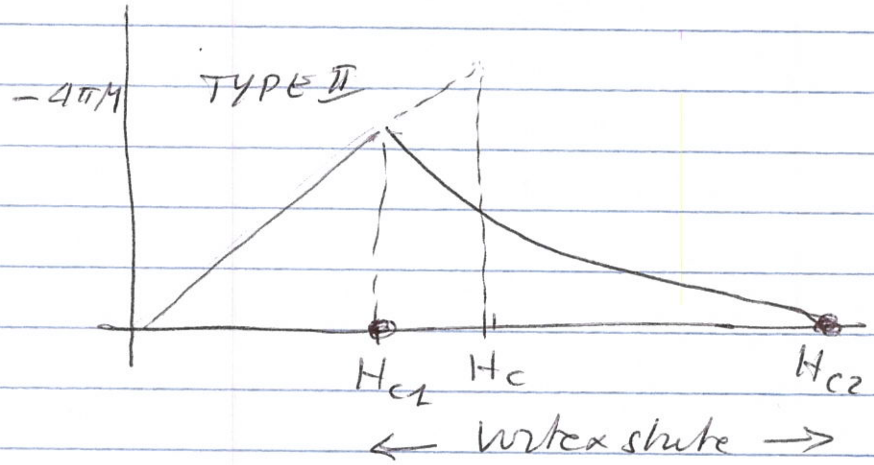
We are going to see that flux is quantized

$$\frac{h}{2e} = 2.07 \times 10^{-15}\ T\,m^2$$

← electron pairs

and each normal core contains just one quantum flux. Devices are very sensitive to small magnetic fields (brain).





In general H_{c2} is $\gg H_c$ (type I) so most applications are with Type II.

Essence of London theory

B_a = applied field
M = magnetization

$B = B_a + 4\pi M = 0$ in SC $\qquad \frac{M}{B_a} = -\frac{1}{4\pi}$

This cannot be derived simply from an ideal conductor with zero resistivity:
Ohm's law $\quad E = \rho j$
at $\rho = 0$ with finite j then $E = 0$
Maxwell:

$$\frac{\partial B}{\partial t} \propto \nabla \times E \rightarrow \rho = 0 \rightarrow \frac{\partial B}{\partial t} = 0$$

and the value of B is frozen.
Remember that this condition leads to a problem of ambiguity of this state because the $\exists$ or not of an internal magnetic field depends on the history of the sample.

We can also write (Newton)

$$E \propto \frac{\partial j}{\partial t}$$

and $\quad \frac{\partial B}{\partial t} \propto \nabla \times \frac{\partial j}{\partial t}$

London ansatz: drop the time derivative

$$B \propto \nabla \times j$$

In the bulk $E = 0 \rightarrow j = 0 \rightarrow B = 0$.
Then analysis of space dependence as before.

Maxwell's equations, vector potential and gauge

$$\begin{cases} \nabla \times H = \frac{\varepsilon}{c}\frac{\partial E}{\partial t} + \underbrace{\frac{4\pi\sigma}{c} E}_{J} \\ \nabla \times E = -\frac{\mu}{c}\frac{\partial H}{\partial t} \\ \nabla \cdot E = \rho_V \\ \nabla \cdot B = 0 \end{cases}$$

Vector potential $A \longrightarrow (E, B) \rightarrow (V, A)$

$$\nabla \times A \rightarrow B$$

$$B = \nabla \times A \; ; \; E = -\nabla \Phi - \frac{\partial A}{\partial t}$$

NB: Gauge: adding a curl free component does not change B

Quantum theory of electrons in a magnetic field

Classically: Lorentz force

$$\vec{F} = q(\vec{E} + \vec{V} \times \vec{B}) \quad (q = -e)$$

Define classical action from Lagrangian

$$S[q_i] = dt\, L(q_i, \dot{q}_i)$$

For conservative forces $L = T - V$

Hamilton's extremal principle:

Trajectories which minimize the action specify the classical equation of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad ; \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

$$H(q_i, p_i) = \sum_i p_i \dot{q}_i - L(q_i, \dot{q}_i)$$

$$B = \nabla \times A \quad ; \quad E = -\nabla\phi - \dot{A}$$

The corresponding Lagrangian is

$$L = \frac{1}{2}mv^2 - q\phi + q\bar{v}\cdot\bar{A} \quad ; \quad q_i = x_i$$

Canonical momentum

$$p_i = \frac{\partial L}{\partial \dot{x}_i} = mv_i + qA_i \quad \text{NB: extra term}$$

then

$$H(q_i, p_i) = \sum_i (mv_i + qA_i)v_i - \frac{1}{2}mv^2 + q\phi -$$
$$- qvA = \frac{1}{2}mv^2 + q\phi$$

But in terms of coordinates and canonical momenta → eq. of motion

$$H = \frac{1}{2m}(\bar{p} - q\bar{A}(r,t))^2 + q\cdot\phi(r,t)$$

$$\dot{x}_i = \frac{\partial H}{\partial p_i} \quad ; \quad \dot{p}_i = -\frac{\partial H}{\partial x_i}$$

NB: $\frac{dp_i}{dt}$ is not the acceleration because A term also varies in time in a complex way.

$$\dot{p}_i = m\ddot{x}_i + q\dot{A}_i = m\ddot{x}_i + q\left(\partial_t A_i + \sum_j v_j \partial_{x_j} A_i\right)$$

$$\dot{p}_i = -\frac{\partial H}{\partial x_i} = \frac{1}{m}\left(\bar{p} - q\bar{A}(r,t)\right) q\, \partial_{x_i}\bar{A} - q\,\partial x_i \phi(r,t)$$

$$= q v_j \partial_{x_i} A_j - q\partial_{x_i}\phi \qquad (\text{check } *)$$

From which the eqs of motion are

$$m\ddot{x}_i = -q\left(\partial_t A_i + v_j \partial_{x_j} A_i\right)$$

and with the suitable transformation this corresponds to Lorentz's equation $\quad m\ddot{x} = \bar{F} = q(\bar{E} + \bar{v}\times\bar{B})$.

Quantum treatment

Canonical quantization procedure

$\hat{p} = -i\hbar\nabla$ so that $[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$

but in this case $\hat{p}_i \neq m\hat{v}_i$ and velocities in different directions do not commute.

NB: With $m\vec{v}_i = -i\hbar\partial_{x_i} - qA_i$ one gets

$$[\hat{v}_x, \hat{v}_y] = \frac{i\hbar q}{m^2} B$$

$$\hat{H} = \frac{1}{2m}\left(\hat{p} - qA(r,t)\right)^2 + q\phi(r,t)$$

The cross term is (paramagnetic term)

$$-\frac{q\hbar}{2im}\left(\nabla A + A\cdot\nabla\right) = \frac{iq\hbar}{m} A\cdot\nabla$$

because of the Coulomb gauge condition $\nabla\cdot A = 0$

Gauge problem

E and B contain only physical degrees of freedom i.e. measurable effects.

$$E = -\nabla\phi - \frac{\partial A}{\partial t} \quad ; \quad B = \nabla \times A$$

Note that if

$$A \to A + \nabla\cdot\Lambda \qquad (\Lambda \text{ is a scalar function})$$

B remains unchanged since

$$B = \nabla\times[A + \nabla\cdot\Lambda] = \nabla\times A$$

since $\nabla\times\nabla\cdot\Lambda = 0$ (rot div A = 0)
rot $\nabla\Lambda = 0$

However this transformation changes E as

$$E = -\nabla\phi - \partial_t A - \nabla\partial_t\Lambda = -\nabla[\phi + \partial_t\Lambda] - \partial_t A$$

But if $\phi \to \phi - \partial_t\Lambda$ the result is unchanged

Therefore E and B are unchanged under the transformation

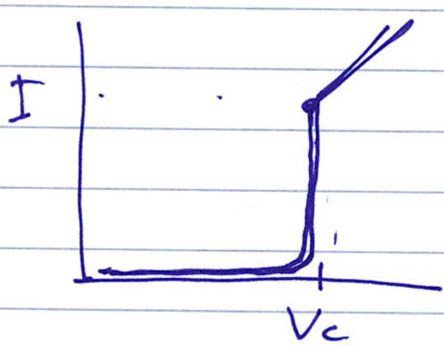
$$\begin{cases} A \to A + \nabla\cdot\Lambda \\ \phi \to \phi - \partial_t\Lambda \end{cases}$$

Gauge is the particular choice of the scalar and vector potential and Λ is the gauge function
The arbitrary numbers of the gauge function correspond to U(1) gauge freedom of the theory

Evidence for a quantum ground state and Macroscopic Coherence

- Zero resistance – persistent current
- Phase transition & specific heat
- Absence of thermoelectric effects
- Ideal diamagnetism $\chi_m = -1$ or $B = 0$.
- Meissner effect
- Flux quantization
- Existence of a gap in the electronic excitations

In a junction SC|Metal|SC one observes that ~~below~~ below a certain voltage there is no current. Only above Vc the pairs are broken and a normal current flows



This is a strong indication that supercurrent are determined by a macroscopic quantum state with a gap for single electron excitations.

London Theory: quantum perspective

From the phenomenology it is suggested that the Supercurrent (at $T=0$) is a property of the quantum ground state. Electrically charged (q) bosonic field which condenses in the g. state into a macroscopic amplitude

$$n_B = |\Psi|^2$$

Em. field (E, B) described with potentials (U, A)

$$\begin{cases} E = -\dfrac{\partial A}{\partial t} - \dfrac{\partial U}{\partial r} \\ B = \dfrac{\partial}{\partial r} \times A \end{cases}$$

E, B are total local fields

$$\frac{1}{2m_B}\left(\frac{\hbar}{i}\frac{\partial}{\partial r} - qA\right)^2 \Psi + qU\Psi = (E - \mu_B)\Psi$$

Energy is measured from the chemical potential of the local field μ_B. In a voltmeter one measures the electrochemical potential

$$\phi = \mu_B + qU$$

then the external potential U, or the effective electric field E_{eff}

$$E_{eff} = -\frac{\partial A}{\partial t} - \frac{1}{q}\frac{\partial \phi}{\partial r}$$

$\left(-i\hbar\dfrac{\partial}{\partial r}\right)$ canonical momentum

$\left(-i\hbar\dfrac{\partial}{\partial r} - qA\right)$ mechanical momentum

Supercurrent density is then

$$J_s = q \frac{p_m}{m_B} n_B = \frac{q^2}{m_B} R(\psi^* p_m \psi) =$$

$$= -\frac{iq\hbar}{2m_B}\left(\psi^* \frac{\partial}{\partial r}\psi - \psi \frac{\partial}{\partial r}\psi^*\right) - \frac{q^2}{m_B}\psi^*\psi A$$

First term : paramagnetic current

Second term : diamagnetic current

<u>NB</u>: Only formal difference because they depend on the gauge.

In a homogeneous SC

$$\psi(r,t) = \sqrt{n_B}\, e^{i\theta(r,t)}$$

$$\Lambda J_s = \frac{\hbar}{q}\frac{\partial\theta}{\partial r} - A \qquad \Lambda = \frac{m_B}{n_B q^2}$$

In the ground state $E = \phi$ and $E\psi = i\hbar\frac{\partial\psi}{\partial t}$

$$\hbar\frac{\partial\theta}{\partial t} = -\phi$$

London limit n_B = const. in space.

$$\frac{\partial}{\partial t}(\Lambda J_s) = -\frac{\partial A}{\partial t} - \frac{1}{q}\frac{\partial\theta}{\partial r} = E_{eff}$$

$$\boxed{\frac{\partial}{\partial t}(\Lambda J_s) = E_{eff}}$$

First London equation. A supercurrent is freely accelerated by an applied voltage.

Or in a bulk SC with no or stationary SCurrent there is no effective electric field. Constant μ_B -

Consider now to apply $\nabla\times$ to the previous equation

$$\nabla\times(\Lambda j_s) = \nabla\times\left[\left(\frac{\hbar}{q}\frac{\partial\theta}{\partial r}\right) - A\right]$$

and consider that $\nabla\times\nabla\theta = 0$

and $B = \nabla\times A$

we obtain

$$\boxed{\nabla\times(\Lambda J_s) = -B}$$

which is the second London equation leading to the Meissner effect etc.

In summary we have seen that the assumption of a macroscopic bosonic ground state and a quantum mechanical analysis leads to the London equations.

Coherence Length ξ

We have defined the penetration depth λ_L from the "local" London equations. In fact they relate the current density at a point $\vec{r}$ with the vector potential at the same point. The ~~coherence length~~ is different. It ~~relates~~ refers to the distance within which the gap cannot change drastically in a varying $B(\vec{r})$ or the range over which we should average A to get j, or is the minimum spatial extent between normal metal and SC.

A spatial variation in the electronic (occupation) states requires kinetic energy. Spatial variations of $j(r)$ should not exceed the stabilization energy of the SC state (gap)

$\psi(r)$

small k

large k, large E(k)

Compare a plane wave $\psi(x) \simeq e^{ikx}$
with the modulated wavefunction

$$\varphi(x) = \frac{1}{\sqrt{2}}\left(e^{i(k+q)x} + e^{ikx}\right)$$

$$\psi^*(x)\,\psi(x) = 1 = \text{constant density}$$

but $\varphi^*\varphi = \frac{1}{2}\left(e^{-i(k+q)x} + e^{-ikx}\right)\left(e^{i(k+q)x} + e^{ikx}\right) =$

$$= \frac{1}{2}\left(2 + e^{iqx} + e^{-iqx}\right) = 1 + \cos qx$$

which is a modulated density with wavevector q

The kinetic energy of $\psi(x)$ is $\hbar^2 k^2/2m$
But for the modulated state

$$\int dx\, \varphi^* \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right)\varphi = \frac{1}{2}\left(\frac{\hbar^2}{2m}\right)\left[(k+q)^2 + k^2\right] =$$

$$= \frac{\hbar^2 k^2}{2m} + \frac{\hbar^2}{2m} k \cdot q \qquad (q \ll k,\ \text{neglect } q^2)$$

If $\frac{\hbar^2 k q}{2m} > E_g$ (gap energy) then the SC will be destroyed.

Therefore the maximum modulation that a SC can accept remaining stable is q_0 (defined by E_g)
This corresponds to an intrinsic coherence length of the SC state $\xi_0 = 1/q_0$

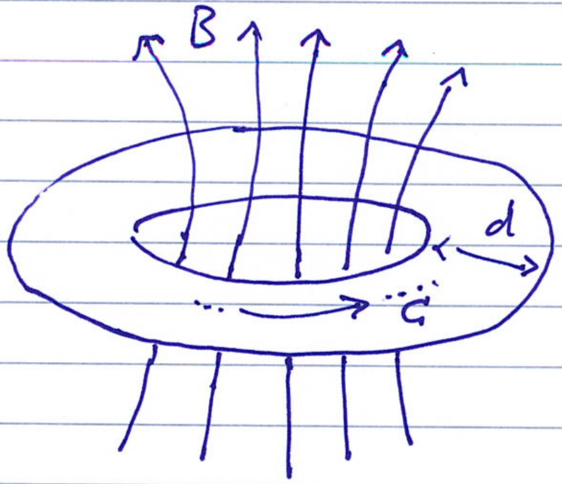
$$\boxed{\xi_0 = \frac{\hbar^2 k_F}{2mE_g} = \frac{\hbar v_F}{2E_g}}$$

Compare this with the London penetration depth λ_L

$$\boxed{\lambda_L = \left(\frac{m}{\mu_0 n e^2}\right)^{1/2}}$$

We can see that λ_L has a classical e.m. origin while ξ_0 has a quantum origin.
Both length are of order 10^{-6} cm but usually $\xi > \lambda_L$ about 10 times. In these cases the London eqs. are a rough approximation of the real situation and their results are not numerically accurate.
The coherence length ξ_0 will follow also from Landau Ginzburg ~~eq~~ and the microscopic BCS analysis of the SC state.
For impure metals both depend strongly on the mean free path l.

Quantization of the Flux in a SC ring



$d \gg \lambda$

Any external field is screened to zero inside a bulk SC state within a surface layer of thickness λ.

Consider a SC ring with magnetic flux Φ passing through it. From the London eqs. $j_s = 0$ deep inside the ring. $\rightarrow$ $E_{eff} = E = 0$ also.

Faraday $\quad \nabla \times E = -\partial B/\partial t$

$$\frac{d\Phi}{dt} = \frac{d}{dt}\int_A B\, dS = -\oint_C E \cdot dl = 0$$

↑ surface defined by C

Therefore even if the supercurrent in a surface layer of the ring is changing with time (variable applied field) the flux Φ is not. The flux through a SC ring is trapped.

Considering the previous result $\Lambda j_s = \frac{\hbar}{q}\frac{\partial \theta}{\partial r} - A$ and integrating along C

$$\oint_C (A + \Lambda j_s) \cdot dl = \frac{\hbar}{q}\oint_C \frac{\partial\theta}{\partial r} \cdot dl$$

But the total change of phase θ of the wavefunction ~~must be~~ around C must be an integer multiple of 2π since the wavefunction must be unique

$$\oint_C (A + \Lambda j_s) \cdot dl = \frac{\hbar}{q} 2\pi n$$

The left hand side was called Fluxoid by F. London
For the case of our ring we have

$$\boxed{\Phi = \frac{\hbar}{q} 2\pi n} = \oint_C (A + \Lambda J_s) \cdot dl$$

By directly measuring the flux quantum Φ_0 one can get that $|q| = 2e$ and

$$\boxed{\Phi_0 = \frac{h}{2e}}$$

If the supercurrent along C is non-zero, the flux Φ is not quantised any more, but the fluxoid is always quantised.

In order to determine the sign of q consider a SC sample which rotates with angular velocity ω.
The sample is neutral so the SC charge density $q n_B$ is neutralised by the charge density $-q n_B$ of the rest of the material.
In the absence of normal current J Ampere's law gives

$$\nabla \times B = \mu_0 (J_s - q n_B v) \; ; \; v = \omega \times r$$

and J_s is the supercurrent with respect to the rest coordinates. Taking the curl and considering

$$\nabla \times v = \nabla \times (\omega \times r) = \underbrace{\omega \frac{\partial r}{\partial r}}_{\substack{\omega \cdot \nabla r \\ \text{three components}}} - \underbrace{(\omega \cdot \nabla) \cdot r}_{\text{only one component}} = 3 - 1 = 2$$

leads to

$$-\nabla^2 B = \mu_0 \nabla \times J_s - 2 \mu_0 q n_B \omega$$

We can now define the London field

$$B_L = -2\lambda_L^2 \mu_0 q n_B \omega = -\frac{2 m_B}{q}\omega$$

and considering the second London eq. ($\nabla \times \Lambda J_s = -B$) we can write

$$\nabla^2 B = \frac{B - B_L}{\lambda_L^2}$$

Deep inside a rotating SC the magnetic field is $\neq 0$ and equal to the homogeneous London field. Independent measurements of the flux quantum and the London field leads to

$$\boxed{q = -2e \quad ; \quad m_B = 2m_e}$$

Therefore the bosonic field is composed of pairs of electrons

More on the flux quantization in a SC ring

Consider a bosonic field (like a large # of photons)
where $n(r)$ is the # of bosons with energy ω
Semiclassical approximation: classical trajectories + phase θ
$E(r)$ = prob. field amplitude

$$E(r) = (4\pi\hbar\omega)^{1/2}\, n(r)^{1/2}\, e^{i\theta(r)}$$

$$E^+(r)\,E(r) = 4\pi(\hbar\omega)\, n(r)$$

For a boson gas we have a large number of bosons in the same orbital. Amplitude $\simeq$ classical object
Amplitude and phase are both observables. Interference.

$$\psi = n^{1/2} e^{i\theta(r)}, \quad \psi^* = n^{1/2} e^{-i\theta(r)} \qquad n = \tfrac{1}{2} n_{\text{electrons}}$$

$$v = \frac{1}{m}\left(p - \frac{q}{c}A\right) = \frac{1}{m}\left(-i\hbar\nabla - \frac{q}{c}A\right)$$

and the particle flux is $\psi^* v \psi = \frac{n}{m}\left(\hbar\nabla\theta - \frac{q}{c}A\right)$

Electric current density $= j = q\psi^* v\psi$

Taking the curl

$$\nabla\times j = -\frac{nq^2}{mc}B \quad \text{(London eq.)}$$

$(N.B.\ \nabla\times\nabla\theta = 0)$

The flux ~~through~~ through the ring is $\Phi = \Phi_{ext} + \phi_{sc}$
Quantization of flux is a consequence of the structure of j.
Consider a closed path C inside the material.
Meissner $B=0$, $j=0$ in the interior

$$\hbar c\,\nabla\theta = qA$$

Consider $\oint_C \nabla\theta \cdot dl = \theta_2 - \theta_1 = 2\pi s$

Stokes theorem

$$\oint_C A \cdot dl = \int_S (\nabla \times A)\, d\sigma = \int_S B \cdot d\sigma = \Phi$$

$$\rightarrow \quad \Phi = \frac{2\pi\hbar c}{q} \cdot s$$

NB: This result does not hold if the flux penetrates the ring

$$\Phi = \Phi_{ext} + \Phi_{sc} \text{ is quantized (the total)}$$

Φ_{ext} is arbitrary so Φ_{sc} has to adjust

Dirac monopole

If a monopole (magnetic) of strength g is located just below the center of the SC ring - The magnetic flux through the ring is $\left(\frac{g}{r^2}\right)(2\pi r^2) = 2\pi g$

and this must be a multiple of $\frac{\pi\hbar c}{e}$

The minimum permissible value of g is then

$$g_{min} = \frac{\hbar c}{2e}$$

NB: if $g < g_{min} \rightarrow J_s$ could compensate

Duration of persistent currents

Consider a persistent current in a ring of SC Type I.
Length L and area A.
The current maintains a flux of a given n fluxoids
A fluxoid can only leave ~~p~~ if a thermal fluctuation turns a minimum volume of the SC ring in the normal state. Prob. per unit of time

$$P \simeq e^{-\Delta F / k_B T}$$

ΔF = (minimum volume) (excess free energy density of normal state)
Minimum volume $\quad V \sim R \xi^2$

Excess free energy $\quad \dfrac{H_c^2}{8\pi}$

$$\Delta F \simeq R \xi^2 \frac{H_c^2}{8\pi}$$

$R = 10^{-4}\,\text{cm}$; $\xi \simeq 10^{-4}\,\text{cm}$, $H_c = 10^3\,\text{G}$

$\rightarrow \Delta F \sim 10^{-7}\,\text{erg}$

For $T = 0.8 T_c$

$$\exp(-\Delta F / k_B T) \simeq \exp(-10^8)$$

Frequency change attempt $\Gamma = \dfrac{E_g}{\hbar}$

$E_g \sim 10^{-15}\,\text{erg} \rightarrow \Gamma \simeq 10^{12}\,\text{sec}^{-1}$

The leakage probability is then

$$P = 10^{12} \times 10^{-4.3\times 10^7} \simeq 10^{-4.3\times 10^7}\,\text{sec}^{-1}$$

$T = 1/P = 10^{4.3\times 10^7}\,\text{sec}$ – Time of the universe $10^{18}\,\text{sec}$

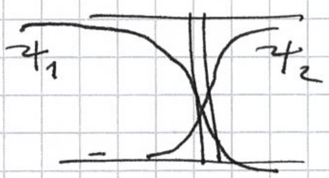
Things may change for type II or very close to T_c

Josephson Superconductor Tunneling

$\boxed{SC \mid Ins \mid SC}$ Remarkable effects

- <u>Dc Josephson effect</u> - A dc current flows across the sample in the absence of any electric or magnetic fields
- <u>Ac Josephson effect</u> - A dc voltage causes rf oscillations across the junction - Precision determination of $\hbar/e$ - Further an rf voltage applied with the dc voltage can cause a dc current across the junction.
- <u>Macroscopic long range quantum interference</u>
  A dc magnetic field applied through a SC circuit containing two junctions causes the maximum supercurrent to show interference effects as a function of B intensity

Dc ~~Joseph~~ Josephson effect



Flux quantization

ψ_1 = prob. amplitude of el. pairs on the side of the junction and ψ_2 on the other side

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi$$

$$i\hbar \frac{\partial \psi_1}{\partial t} = \hbar T \psi_2 \quad ; \quad i\hbar \frac{\partial \psi_2}{\partial t} = \hbar T \psi_1$$

$\hbar T$ is the transfer interaction

$$\psi_1 = n_1^{1/2} e^{i\theta_1} \quad ; \quad \psi_2 = n_2^{1/2} e^{i\theta_2}$$

$$\begin{cases} \dfrac{\partial \psi_1}{\partial t} = \dfrac{1}{2} n_1^{-1/2} e^{i\theta_1} \dfrac{\partial n_1}{\partial t} + i\psi_1 \dfrac{\partial \theta_1}{\partial t} = -iT\psi_2 \\[2ex] \dfrac{\partial \psi_2}{\partial t} = \dfrac{1}{2} n_2^{-1/2} e^{i\theta_2} \dfrac{\partial n_2}{\partial t} + i\psi_2 \dfrac{\partial \theta_2}{\partial t} = -iT\psi_1 \end{cases}$$

Multiplying the first equation by $n_1^{1/2} e^{i\theta_1}$
with $\delta = \theta_2 - \theta_1$

$$\frac{1}{2}\frac{\partial n_1}{\partial t} + i n_1 \frac{\partial \theta_1}{\partial t} = -iT\,(n_1 \cdot n_2)^{1/2} e^{i\delta}$$

Multiplying the second equation by $n_2^{1/2} e^{-i\theta_2}$

$$\frac{1}{2}\frac{\partial n_2}{\partial t} + i n_2 \frac{\partial \theta_2}{\partial t} = -iT\,(n_1 \cdot n_2)^{1/2} e^{-i\delta}$$

Now equate real and imaginary part of both eqs.

$$\frac{\partial n_1}{\partial t} = 2T(n_1 n_2)^{1/2} \sin\delta \quad ; \quad \frac{\partial n_2}{\partial t} = -2T(n_1 n_2)^{1/2} \sin\delta$$

$$\frac{\partial \theta_1}{\partial t} = -T\left(\frac{n_2}{n_1}\right)^{1/2} \cos\delta \quad ; \quad \frac{\partial \theta_2}{\partial t} = -T\left(\frac{n_1}{n_2}\right)^{1/2} \cos\delta$$

Taking $n_1 \simeq n_2$ we have

$$\frac{\partial \theta_1}{\partial t} = \frac{\partial \theta_2}{\partial t} \quad ; \quad \frac{\partial}{\partial t}(\theta_2 - \theta_1) = 0$$

and also

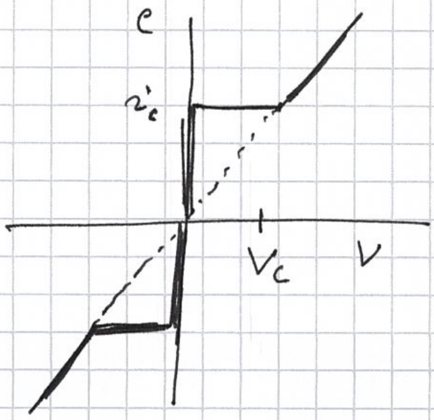
$$\frac{\partial n_2}{\partial t} = -\frac{\partial n_1}{\partial t}$$

But also $J_{12} \sim \frac{\partial n_2}{\partial t}$ or $-\frac{\partial n_1}{\partial t}$

and therefore

$$\boxed{J = J_0 \sin\delta = J_0 \sin(\theta_2 - \theta_1)}$$

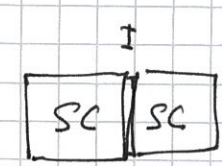
$J_0 \propto T$. The current J_0 is the maximum zero voltage current possible. So with no applied voltage a dc current will flow across the junction with a value between $-J_0$ and J_0 depending on $\theta_2 - \theta_1$



This is the Josephson effect.

[By increasing the polarisation current the voltage difference remain zero up to a critical current]

AC Josephson Effect



Applying a voltage V through the junction

Energy difference for a pair to cross is qV $\quad q = -2e$

Assign a potential energy $-eV$ on one side and $+eV$ on the other side.

Equations of motion:

$$i\hbar \frac{\partial \Psi_1}{\partial t} = \hbar T \Psi_2 - eV\Psi_1 \quad ; \quad i\hbar \frac{\partial \Psi_2}{\partial t} = \hbar T \Psi_1 + eV \Psi_2$$

(Proceeding as before

$$\frac{1}{2}\frac{\partial n_1}{\partial t} + i n_1 \frac{\partial \theta_1}{\partial t} = -i T (n_1 n_2)^{1/2} e^{i\delta} \quad)$$

we get now

$$\frac{1}{2}\frac{\partial n_1}{\partial t} + i n_1 \frac{\partial \theta_1}{\partial t} = \underline{i e V n_1 \hbar^{-1}} - i T (n_1 n_2)^{1/2} e^{i\delta}$$

The real part is

$$\frac{\partial n_1}{\partial t} = 2T(n_1 n_2)^{1/2} \sin\delta \qquad (\text{like before } V=0)$$

and imaginary part

$$\Longrightarrow \frac{\partial \theta_1}{\partial t} = \underline{\underline{(eV/\hbar)}} - T(n_2/n_1)^{1/2}\cos\delta$$

and from the second equation

$$\frac{1}{2}\frac{\partial n_2}{\partial t} + i n_2 \frac{\partial \theta_2}{\partial t} = -i e V n_2 \hbar^{-1} - i T (n_1 n_2)^{1/2} e^{-i\delta}$$

and

$$\frac{\partial n_2}{\partial t} = -2T(n_1 n_2)^{1/2} \sin\delta$$

$$\longrightarrow \frac{\partial \theta_2}{\partial t} = -(eV/\hbar) - T(n_1/n_2)^{1/2}\cos\delta$$

and taking $n_1 \simeq n_2$

$$\frac{\partial(\theta_2 - \theta_1)}{\partial t} = \frac{\partial \delta}{\partial t} = -2eV/\hbar$$

Therefore the relative phase varies as

$$\delta(t) = \delta(0) - (2eVt/\hbar)$$

The current is therefore

$$\boxed{J = J_0 \sin\left[\delta(0) - 2eVt/\hbar\right]}$$ Ac Josephson

and the current oscillates with a ~~phase~~ frequency $\omega = \frac{2eV}{\hbar}$

A dc voltage of $1\,\mu V \rightarrow \omega = 483\ MHz$

The frequency can be interpreted that a photon of energy $\hbar\omega = 2eV$ is emitted or absorbed when an electron pair crosses the barrier

* By measuring voltage and frequency → accurate measure of $\frac{e}{\hbar}$ -

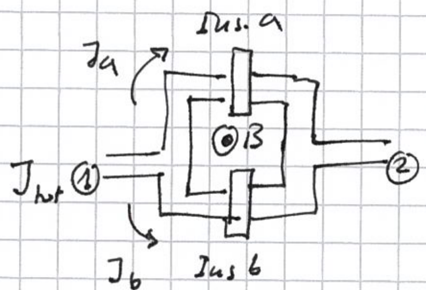
Macroscopic Quantum Interference (SQUID)

The phase difference along a closed circuit is

$$\theta_2 - \theta_1 = (2e/\hbar c)\,\Phi$$

and the flux is both external and due to currents.

Consider two Josephson junctions in parallel



Let the phase difference along the path (a) be δ_a and in b is δ_b

In the absence of B these phases are equal

Now let a flux Φ pass through the interior of the circuit (with a solenoid)

We have

check

$$\delta_b = \delta_a - \left(\frac{2e}{\hbar c}\right)\Phi$$

$$\delta_b = \delta_0 + \frac{e}{\hbar c}\Phi \quad ; \quad \delta_a = \delta_0 - \frac{e}{\hbar c}\Phi$$

Total current is $J_a + J_b$

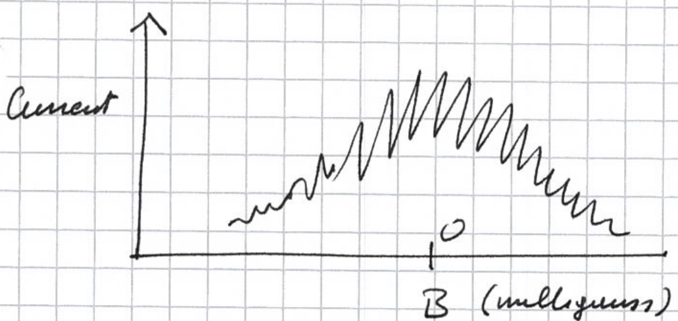
and from J. effect

$$J_{total} = J_0 \left\{ \sin\left(\delta_0 + \frac{e}{\hbar c}\Phi\right) + \sin\left(\delta_0 - \frac{e}{\hbar c}\Phi\right) \right\} =$$

$$= 2\left(J_0 \sin\delta_0\right) \cos\frac{e\Phi}{\hbar c}$$

The current varies with Φ and has maxima when

$$\frac{e\Phi}{\hbar c} = s\pi \qquad s = \text{integer}$$

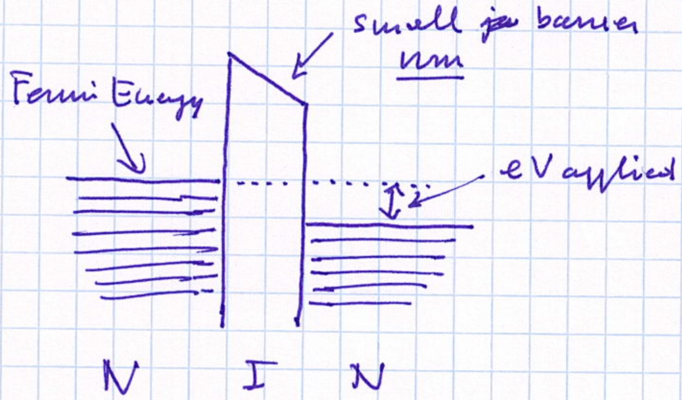


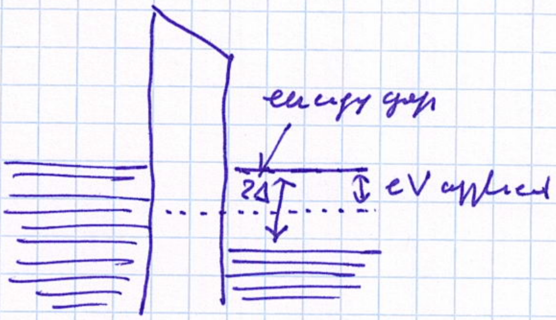
Shorter variation is the interference
Longer variation is a diffraction effect due to the size of the junction

Josephson Effect

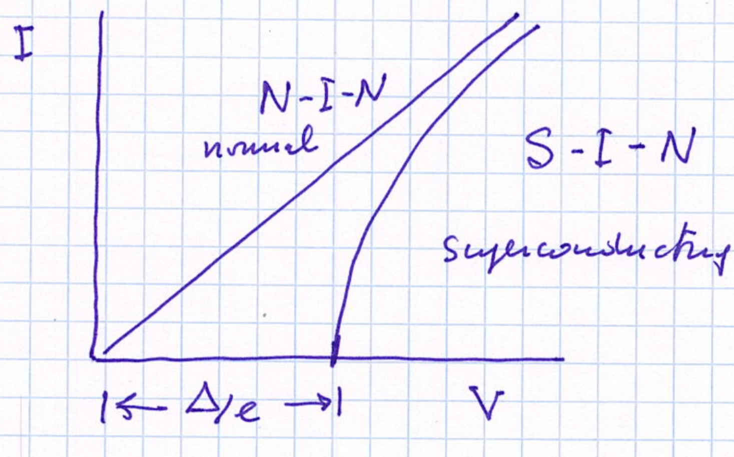
B. Josephson 1962

Prediction that in a junction SC|I|SC a ~~curr~~ free supercurrent can flow even at $V=0$.

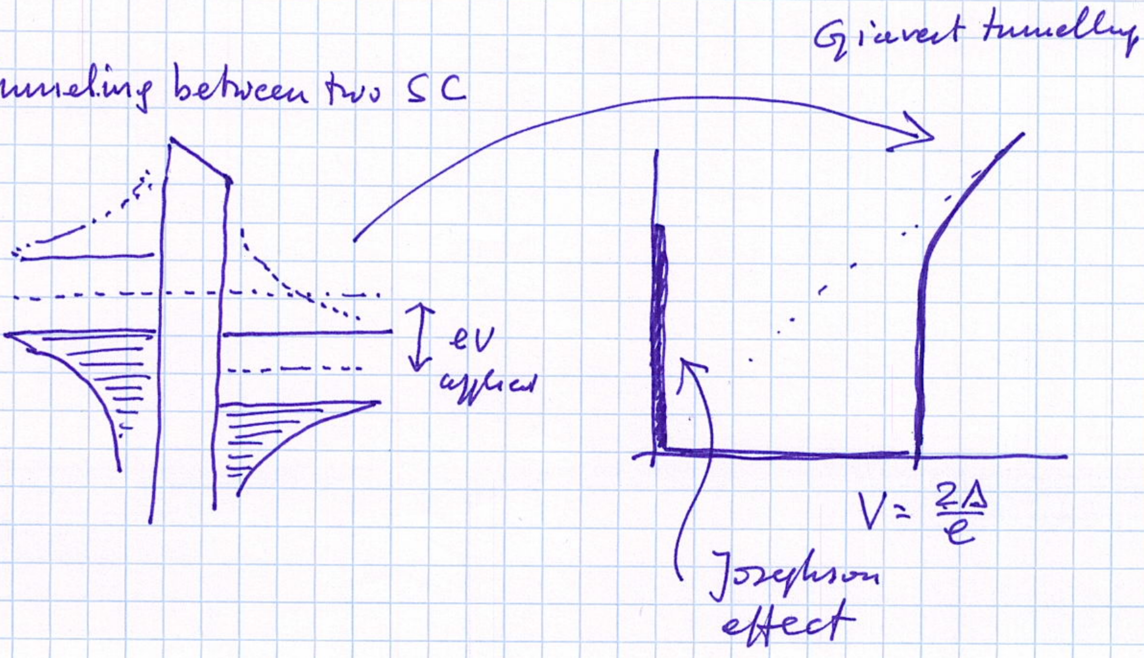




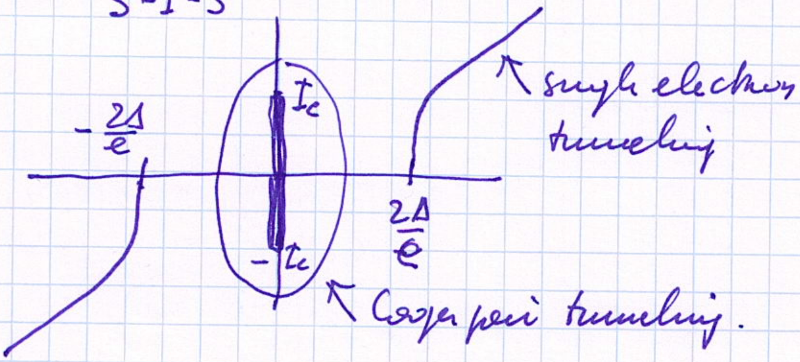
Single particle tunnelling through the insulator
→ Normal resistance → Ohm's law



Tunneling between two SC



S-I-S



For the supercurrent we can write

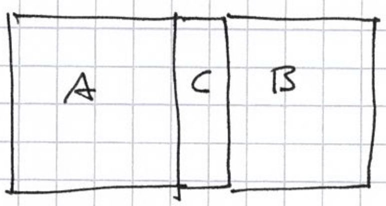
$$J_s(r,t) = -\frac{1}{\Lambda}\left(A(r,t) + \frac{\Phi_0}{2\pi}\nabla\theta(r,t)\right)$$

$$\frac{\partial \theta(r,t)}{\partial t} = -\frac{1}{\hbar}\left(\frac{\Lambda J_s^2}{2n^*} + q^*\phi(r,t)\right)$$

For comparison:

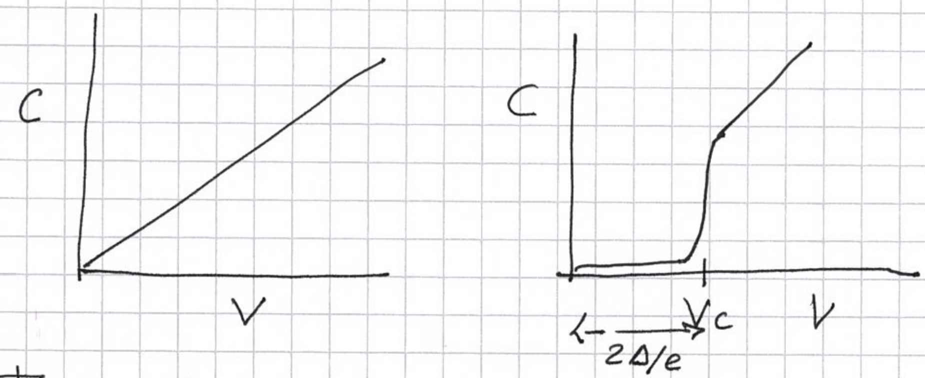
Single Particle Tunnelling

Consider two metals separated by a thin insulator.

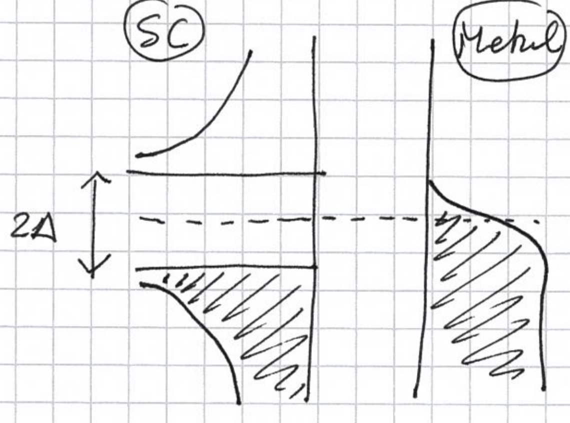


If C is very thin 10-20 Å an electron can go through the barrier = tunneling

When both metals are normal conductors C-V relation is directly proportional to the voltage. But if one of the metals is SC the CV characteristic changes a lot (Giaever)



This is because in a SC there is an energy gap that the electrons have to overcome



At $T=0$ no current can flow unless $V = \frac{E_g}{2e} = \frac{\Delta}{e}$

E_g = break up E of the electron pair

At finite T there is a small current also at small V.

NB: Il prossimo capitolo inizia con la pagina 33 perché quest'anno ho riscritto questa prima parte che è venuta un po' più lunga.